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On a characterization of algebraic number fields by their Galois groups of p-closed Galois extensions

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In this note, we give a characterization of finite algebraic number fields by the Galois groups of their p-closed Galois extensions. This characterization is a refinement of a theorem of K. Uchida [11]. For details, see [9].

We use the following notations throughout this note.

Notations. Let K be a field of characteristic 0 and let p be a prime number. For a normal extension L/K, $G(L/K)$ denotes its Galois group. In this note, a "p-extension" always means a normal p-extension. A "solvable" extension is a normal extension whose Galois group is a projective limit of finite solvable groups.

$\bar{K}$: the algebraic closure of K,

$\tilde{K}$: the solvable closure of K (i.e. the maximal solvable extension over K),

$K(p)$: the maximal p-extension over K,

$G_K = G(\bar{K}/K)$: the absolute Galois group of K,

$\tilde{G}_K = G(\tilde{K}/K)$, $G_K(p) = G(K(p)/K)$,

ζ_p: a primitive p-th root of unity in $\overline{K}$,

P_0: the set of all prime numbers.

§1. Introduction.

Let k_1 and k_2 be finite algebraic number fields. In 1969, J. Neukirch characterized finite normal algebraic number fields by their absolute Galois groups.

THEOREM A (Neukirch [5]). If $k_1/\mathbb{Q}$ is normal and $G_{k_1} \simeq G_{k_2}$, then $k_1 = k_2$.

And he conjectured [5]:

If $G_{k_1} \simeq G_{k_2}$, then $k_1 \simeq k_2$.

Furthermore, he proved a refinement of Theorem A.

THEOREM A′ (Neukirch [6]). If $k_1/\mathbb{Q}$ is normal and $\tilde{G}_{k_1} \simeq \tilde{G}_{k_2}$, then $k_1 = k_2$.

Neukirch's conjecture was proved by Uchida [10], [11], in a generalized form.

THEOREM B (Uchida [11]). Let Ω_1/k_1 and Ω_2/k_2 be solvably closed (i.e. Ω_1 and Ω_2 have no proper abelian extension) Galois extensions. If there exists a topological isomorphism σ: $G(\Omega_1/k_1) \tilde{\to} G(\Omega_2/k_2)$, then there exists a unique isomorphism of fields g: $\Omega_1 \tilde{\to} \Omega_2$ such that $\sigma(h) = ghg^{-1}$ for all $h \in G(\Omega_1/k_1)$. In particular, $g|_{k_1}$ gives an isomorphism of fields k_1 and k_2.

In this note, we consider the following problem.

PROBLEM. In Theorem B, can we replace Ω_1/k_1 and Ω_2/k_2 with some smaller extensions?

We give an answer to this problem by using p-closed extensions.

To prove Theorems A and A′, Neukirch used a characterization of algebraic number fields with henselian valuations [5], [6]. So, first, we generalize his characterization in §2, and next, we apply it to finite algebraic number fields in §3.

§2. $\tilde{p}$-closed extensions and Ω-henselian fields.

Let Ω be a field of characteristic 0, p be a prime number and P be a subset of P_0.

DEFINITION. We call Ω $\tilde{p}$-closed if and only if Ω is p-closed (i.e. Ω has no proper p-extension) and Ω contains ζ_p. We call Ω <u>$\tilde{P}$-closed</u> if and only if Ω is $\tilde{p}$-closed for all $p \in P$.

<u>REMARK</u> 1. Ω is solvably closed if and only if Ω is $\tilde{P}_0$-closed.

<u>REMARK</u> 2. Let K be a field of characteristic 0 and let P be a subset of P_0. We put $K(\tilde{P}) = \bigcup_{i=0}^{\infty} K_i$, where

$$\begin{cases} K_0 = \coprod_{p \in P} K(\zeta_p): \text{ the composite field of } K(\zeta_p),\ p \in P, \\ K_{i+1} = \coprod_{p \in P} K_i(p): \text{ the composite field of } K_i(p),\ p \in P \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 0, 1, 2, \cdots). \end{cases}$$

Then, $K(\tilde{P})$ is the minimal $\tilde{P}$-closed Galois extension over K. If k is a finite algebraic number field and $P \subsetneq P_0$, then $k(\tilde{P}) \subsetneq \tilde{k}$.

Now, let K be an algebraic number field (not necessarily finite over $\mathbb{Q}$) and $v|\ell$ be a valuation of K induced from a fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_\ell$, where ℓ is either a prime number or ∞ and $\mathbb{Q}_\infty$ denotes $\mathbb{R}$. We put $K_v = K \cdot \mathbb{Q}_\ell$. Let Ω/K be an algebraic extension.

<u>DEFINITION</u>. We call K <u>Ω-henselian</u> with respect to v if and only if there exists only one extension $\tilde{v}$ of v to Ω

(i.e. for any extension $\Omega \hookrightarrow \overline{\mathbb{Q}}_\ell$ of the embedding $K \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have $\Omega \cap K_v = K$). If K is $\overline{\mathbb{Q}}$-henselian with respect to v, then K is simply called <u>henselian</u> with respect to v.

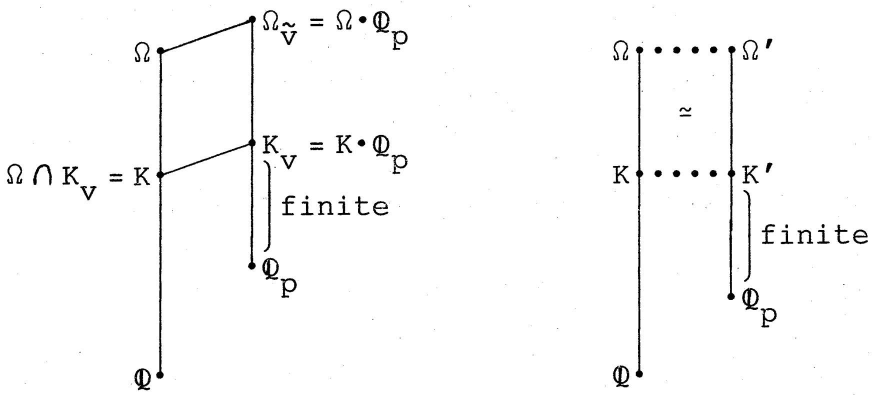
In the case of $\tilde{p}$-closed Galois extensions, we can characterize algebraic number fields which are Ω-henselian with respect to non-archimedean valuations, by their Galois groups.

<u>THEOREM</u> 1. Let p be a prime number and let Ω/K be a $\tilde{p}$-closed (i.e. Ω is $\tilde{p}$-closed) Galois extension of algebraic number fields. Then the following two conditions are equivalent.

(i) There exists a non-archimedian valuation $v|p$ of K such that K is Ω-henselian with respect to v and $[K_v:\mathbb{Q}_p] < \infty$.

(ii) There exist a finite extension $K'/\mathbb{Q}_p$ and a $\tilde{p}$-closed Galois extension Ω'/K' such that $G(\Omega/K) \simeq G(\Omega'/K')$.

Furthermore, v (in (i)) obtained from the condition (ii) is unique and $[K_v:\mathbb{Q}_p] = [K':\mathbb{Q}_p]$ holds.



(i) (ii)

REMARK 3. In the following three cases, Theorem 1 has been proved.

Case 1. $\Omega = \overline{\mathbb{Q}}$ and $\Omega' = \overline{\mathbb{Q}}_p$: by Neukirch [5].

Case 2. $\Omega = \tilde{K}$ and $\Omega' = \tilde{K}'$ $(= \overline{\mathbb{Q}}_p)$: by Neukirch [6].

Case 3. $\zeta_p \in K$ and $\Omega = K(p)$ (hence $\zeta_p \in K'$ and $\Omega' = K'(p)$): by Y. Hironaka-Kobayashi [3].

REMARK 4. Case 1 (in Remark 3) of Theorem 1 is a p-adic analogue of a theorem of E. Artin [1]:

If K $(\neq \overline{\mathbb{Q}})$ is an algebraic number field and $[\overline{\mathbb{Q}}:K]$ is finite, then K is henselian with respect to a unique archimedian valuation of K and $\overline{\mathbb{Q}} = K(\sqrt{-1})$, $[\overline{\mathbb{Q}}:K] = 2$.

We can generalize Artin's theorem as follows:

Let p be a prime number and Ω/K $(\Omega \neq K)$ be a p-closed finite p-extension of algebraic number fields. Then $p = 2$ and K is Ω-henselian with respect to a unique archimedean valuation of K, and $\Omega = K(\sqrt{-1})$, $[\Omega:K] = 2$.

§3. A characterization of finite algebraic number fields.

For a finite algebraic number field k and a prime number p, we put $S_p(k) = \{\mathfrak{p} \mid \text{a prime ideal of } k \text{ above } (p)\}$. For $\mathfrak{p} \in S_p(k)$, we use the following notations.

$k_{\mathfrak{p}}$: the completion of k with respect to $\mathfrak{p}$,

$e(\mathfrak{p}/p)$: the ramification index of $k_{\mathfrak{p}}/\mathbb{Q}_p$,

$f(\mathfrak{p}/p)$: the relative degree of $k_{\mathfrak{p}}/\mathbb{Q}_p$.

Then, from Theorem 1, we obtain the following

COROLLARY. Let p be a prime number, k_1 and k_2 be finite algebraic number fields, and Ω_1/k_1 and Ω_2/k_2 be $\tilde{p}$-closed Galois extensions. If $G(\Omega_1/k_1) \simeq G(\Omega_2/k_2)$, then there exists a bijection $\phi_p\colon S_p(k_1) \to S_p(k_2)$ such that $[k_{\mathfrak{p}}:\mathbb{Q}_p] = [k_{\phi_p(\mathfrak{p})}:\mathbb{Q}_p]$ for all $\mathfrak{p} \in S_p(k_1)$.

PROOF. Using Theorem 1, we can define ϕ_p by the 1-1 correspondence of the decomposition subgroups of the prime ideals above (p) of k_1 and k_2.

Let $A = (r; f_1, \cdots, f_r)$ be a tuple of natural numbers such that $f_1 \leqq \cdots \leqq f_r$. For such A and a finite algebraic number field k, we put

$$P_A(k) = \left\{ p \in P_0 \;\middle|\; \begin{array}{l} (p) = \mathfrak{p}_1^{e(\mathfrak{p}_1/p)} \cdots \mathfrak{p}_r^{e(\mathfrak{p}_r/p)} \quad \text{in } k, \\ f(\mathfrak{p}_i/p) = f_i \quad (1 \leqq i \leqq r). \end{array} \right\}$$

For $P \subset P_0$, we put

$$\delta(P) = \lim_{s \to 1+0} \Big(\sum_{p \in P} \frac{1}{p^s} \Big) / \log \frac{1}{s-1} \quad \text{(if it exists)}, \; 0 \leqq \delta(P) \leqq 1$$

($\delta(P)$ is called the Dirichlet density of P).

For two subsets $P_1, P_2 \subset P_0$, we write

$P_1 \doteqdot P_2$ if and only if $\#((P_1 \cup P_2) - (P_1 \cap P_2)) < \infty$,

$P_1 \underset{\delta}{=} P_2$ if and only if $\delta((P_1 \cup P_2) - (P_1 \cap P_2)) = 0$.

DEFINITION. Let k_1 and k_2 be finite algebraic number fields. Then k_1 and k_2 are called <u>arithmetically equivalent</u> over $\mathbb{Q}$ if and only if $P_A(k_1) \doteqdot P_A(k_2)$ for all $A = (r; f_1, \cdots, f_r)$ (This is equivalent to $P_A(k_1) \underset{\delta}{=} P_A(k_2)$ for all $A = (r; f_1, \cdots, f_r)$. For arithmetically equivalent fields, see e.g. [2], [4], [7]).

<u>THEOREM</u> 2. Let P be a subset of P_0 such that $\delta(P) = 1$. Let k_1 and k_2 be finite algebraic number fields and let Ω_1/k_1 and Ω_2/k_2 be $\tilde{P}$-closed Galois extensions. If there exists a topological isomorphism $\sigma\colon G(\Omega_1/k_1) \xrightarrow{\sim} G(\Omega_2/k_2)$, then there exists a unique isomorphism of fields $g\colon \Omega_1 \xrightarrow{\sim} \Omega_2$ such that $\sigma(h) = ghg^{-1}$ for all $h \in G(\Omega_1/k_1)$. In particular, $g|_{k_1}$ gives an isomorphism of fields k_1 and k_2.

<u>PROOF</u>. From Corollary, it follows easily that k_1 and k_2 are arithmetically equivalent over $\mathbb{Q}$. Let k_1' be an intermediate field of Ω_1/k_1 such that k_1'/k_1 is finite, and let k_2' be the corresponding subfield of Ω_2 by σ, then k_1' and k_2' are also arithmetically equivalent over $\mathbb{Q}$. Using this, we can prove Theorem 2 by slightly modifying the proof of Theorem B.

<u>REMARK</u> 5. In Theorem 2, the conclusion $k_1 \simeq k_2$ (over $\mathbb{Q}$) cannot be strengthened to $k_1 \simeq k_2$ over $k_1 \cap k_2$.

Example. Put $k_1 = \mathbb{Q}(\sqrt[4]{2})$ and $k_2 = \mathbb{Q}(\sqrt[4]{2}\cdot\sqrt{-1})$. Then, $k_1 \cap k_2 = \mathbb{Q}(\sqrt{2})$. Since $k_1 \simeq k_2$ (over $\mathbb{Q}$), $G_{k_1} \simeq G_{k_2}$. But, for any isomorphism $g\colon k_1 \xrightarrow{\sim} k_2$, we have

$g(\sqrt{2}) = -\sqrt{2}$. Hence, g cannot be an isomorphism over $k_1 \cap k_2$.

§4. An outline of the proof of Theorem 1.

Using Krasner's lemma, we can prove the following two lemmas.

LEMMA 1. Let p be a prime number, Ω be a $\tilde{p}$-closed algebraic number field and v be a non-archimedean valuation of Ω. Then Ω_v is also $\tilde{p}$-closed.

LEMMA 2. Let p be a prime number and Ω/K be a $\tilde{p}$-closed Galois extension of algebraic number fields. If $p \mid [\Omega:K]$, then K is Ω-henselian with respect to at most one non-archimedean valuation.

We use the following propositions from Galois cohomology (See [5], [6], [8]).

PROPOSITION 1. Let ℓ, p be prime numbers and $K/\mathbb{Q}_\ell$ be an algebraic extension.

(1) If $p^\infty \nmid [K:\mathbb{Q}_\ell]$ and $\zeta_p \notin K$, then

$$G_K(p) \text{ is a free pro-p-group of rank } \begin{cases} 1 & (\ell \neq p), \\ [K:\mathbb{Q}_p]+1 & (\ell = p) \end{cases}$$

(Here, if $[K:\mathbb{Q}_p] = \infty$, then $[K:\mathbb{Q}_p] + 1$ means $\aleph_0$.), and $cd_p(G_K(p)) = 1$.

(2) If $p^\infty \nmid [K:\mathbb{Q}_\ell]$ and $\zeta_p \in K$, then

$$\begin{cases} \text{generator-rank } (G_K(p)) = \begin{cases} 2 & (\ell \neq p), \\ [K:\mathbb{Q}_p] + 2 & (\ell = p) \end{cases} \\ \quad \text{(Here, if } [K:\mathbb{Q}_p] = \infty, \text{ then } [K:\mathbb{Q}_p]+2 \text{ means } \aleph_0.), \\ \text{relation-rank } (G_K(p)) = 1, \end{cases}$$

and $cd_p(G_K(p)) = 2$.

(3) If $p^\infty | [K:\mathbb{Q}_\ell]$, then $G_K(p)$ is a free pro-p-group and $cd_p(G_K(p)) \leqq 1$.

<u>PROPOSITION</u> 2. Let K be an algebraic number field, then the canonical homomorphism $B_K \xrightarrow{(\mathrm{Res}_v)} \coprod_v B_{K_v}$ is <u>injective</u>. Here, B_K and B_{K_v} denote the Brauer groups of K and K_v, respectively, and v runs over all valuations of K.

<u>An outline of the proof of Theorem 1</u>. First, we assume (i). Let $\tilde{v}$ be the unique extension of v to Ω. We put $K' = K_v$ and $\Omega' = \Omega_{\tilde{v}}$. Then Ω' is $\tilde{p}$-closed by Lemma 1, and $[K':\mathbb{Q}_p]$ is finite by the assumption. Since $\Omega \cap K_v = K$ by the assumption, we have $G(\Omega/K) \simeq G(\Omega'/K')$. Next, we assume (ii). Let $G(\Omega/L)$ be a p-Sylow subgroup of $G(\Omega/K)$ and $G(\Omega'/L')$ be the corresponding p-Sylow subgroup of $G(\Omega'/K')$ by the isomorphism. Then $\Omega = L(p)$, $\zeta_p \in \Omega$, $\Omega' = L'(p)$, $\zeta_p \in L'$ and $p^\infty \nmid [L':\mathbb{Q}_p]$. By Proposition 1, we have $cd_p(G_{L'}(p)) = 2$, therefore $cd_p(G_L(p)) = 2$

and $B_L(p) \neq 0$. Then, by Proposition 2, there exists a non-archimedean valuation w of L (say $w|\ell$) such that $B_{L_w}(p) \neq 0$ i.e. $p^\infty \nmid [L_w:\mathbb{Q}_\ell]$. Let $\bar{w}$ be an extension of w to Ω and put $v = \bar{w}|_K$, then we can prove the following:

$$\begin{cases} p = \ell \text{ (by Proposition 1)}, \\ \bar{w} \text{ is the unique extension of } v \text{ to } \Omega, \\ v \text{ is unique (by Lemma 2)}, \\ [K_v:\mathbb{Q}_p] = [K':\mathbb{Q}_p] < \infty \text{ (by Proposition 1)}. \end{cases}$$

This is an outline of the proof of Theorem 1.

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